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with the Author's kind regards

(602)

ON THE CORRELATION OF TWO PLANES.*

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Definition and Determination of a Correlation.

1. A correlation is said to be established between two planes, when their points and right lines are so associated, that to each point in one of the planes, and to each line passing through that point, respectively correspond, in the other plane, *one* line and *one* point in that line.

2. It will be convenient to apply the familiar terms *pole* and *polar* to a point and its corresponding right line.

3. It has been shown by Chasles† and others, that such a correlation may be established, and that in one way only, when four points, in one plane, and their polars, in the other plane, are given; provided always that no three of the points are collinear, and no three polars concurrent. In fact, if $A_1\,B_1\,C_1\,D_1$ be the four points in the first plane, and $a_2\,b_2\,c_2\,d_2$ their respective polars in the other, then, x_2 being the polar of any fifth point X_1, it follows at once from the definition of a correlation that the pencils $A_1\,(B_1\,C_1\,D_1\,X_1)$ and $B_1\,(A_1\,C_1\,D_1\,X_1)$ must be equi-anharmonic, respectively, with the rows $a_2\,(b_2\,c_2\,d_2\,x_2)$ and $b_2\,(a_2\,c_2\,d_2\,x_2)$. Hence, by a well-known method, if X_1 be given, the intersections of its polar x_2 with a_2 and b_2, and hence x_2 itself, are readily found.

4. From the definition of a correlation it also follows that if the polar of A_1 pass through A_2, the polar of A_2 will pass through A_1; and similarly, that if the pole of a_1 lie on a_2, the pole of a_2 will lie in a_1. Hence we may term A_1 and A_2 *conjugate points*, and a_1 and a_2 *conjugate lines* of the correlation.

5. That in any correlation two given points, or two given right lines,

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* Of the two questions, Correlation and Homography, the first has been selected for special consideration in this paper. With suitable modifications, however, as shown in the foot-notes on pages 22 and 23, the results are all applicable to the second question.

† "Géométrie Supérieure," art. 581.

shall be conjugate to each other, is obviously to be regarded as one condition; that a given point and line shall bear to each other the relation of pole and polar is, accordingly, equivalent to two conditions. Hence, and from art. 3, we may infer that eight conditions are necessary and sufficient for the establishment of a correlation between two planes.*

6. The problem to determine a correlation between two planes which shall satisfy *any* eight given conditions, is susceptible, in general, of a finite number of solutions. The determination of this number, when the conditions are of the elementary kind, single and double, described in the last article,—a necessary step towards the solution of the more general problem,—is one of the objects of the present paper.

Systems of Correlations.

7. Two planes may obviously be correlated in innumerable ways so as to satisfy *seven* given conditions; the totality of such correlations constitute what may be termed *a system of correlations satisfying seven conditions.*

For example, if the polars $p_2\ q_2\ r_2$ of three points $P_1\ Q_1\ R_1$ were given, as well as two conjugate points A_1 and A_2,—a set of seven conditions which may be conveniently indicated thus:

$$\left.\begin{matrix} P_1 & Q_1 & R_1 & A_1 \\ p_2 & q_2 & r_2 & A_2 \end{matrix}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

then, in general, any line a_2 passing through A_2 may be regarded as the polar of A_1, and a correlation established, in the manner indicated in art. 3, which shall satisfy the eight conditions,

$$\begin{matrix} P_1 & Q_1 & R_1 & A_1 \\ p_2 & q_2 & r_2 & a_2 \end{matrix}$$

and therefore, *à fortiori*, the seven given conditions (1).

By giving to a_2 all possible positions, we obtain the several correlations of the system. We shall find it convenient to denote such a system by the symbol $\left(\begin{matrix} P_1 & Q_1 & R_1 & A_1 \\ p_2 & q_2 & r_2 & A_2 \end{matrix}\right)$.

8. It should be observed, however, that whenever a_2 passes through the intersection of any two of the three given lines $p_2\ q_2\ r_2$, the provision alluded to in art. 3 ceases to be observed, and no correlation, in the ordinary sense of the term, can be established.

9. Such exceptions occur in every system of correlations. As

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* This also follows immediately from the "*expression analytique des figures corrélatives*," given by Chasles in the "Géométrie Supérieure," art. 594. Expressed in trilinear co-ordinates, the result there arrived at is that $\lambda_2 x + \mu_2 y + \nu_2 z = 0$ will be the polar of $\alpha_1\beta_1\gamma_1$ if

$$\lambda_2 : \mu_2 : \nu_2 = l\alpha_1 + m\beta_1 + n\gamma_1 : l'\alpha_1 + m'\beta_1 + n'\gamma_1 : l''\alpha_1 + m''\beta_1 + n''\gamma_1,$$

where the ratios of $l, m, n,\ l', m', n',\ l'', m'', n''$ are the eight constants of the correlation.

another example we may take the system

$$\begin{pmatrix} P_1 & Q_1 & R_1 & a_1 \\ p_2 & q_2 & r_2 & a_2 \end{pmatrix},$$

wherein a_1 and a_2 denote given conjugate lines. The several correlations of this system are obviously obtained by giving to A_1, the pole of a_2, all possible positions on a_1; but whenever A_1 comes into line with two of the three points P_1 Q_1 R_1, the method of establishing a correlation described in art. 3 ceases to be applicable.

10. Instead of excluding cases such as those described in the last two articles, it will be found of the highest importance to admit them as *exceptional correlations* into the system satisfying seven conditions, and carefully to study their properties. The part they play in the general theory of correlation will be found to be strictly analogous to that played by degenerate conics in the investigations of Chasles on systems of conics satisfying four conditions.*

Origin and Nature of Exceptional Correlations.

11. With a view of obtaining an insight into the nature and origin of exceptional correlations, we will first consider all the exceptional forms which a homographic relation between two planes may assume; or, what is equally general, all the exceptional modes of putting two planes Π_1, Π' into perspective with each other.

It is clear that, so long as the centre of perspective lies in neither of the two planes, the homographic relation between them is of the ordinary type; we have, therefore, merely to consider the cases where the centre of perspective lies, 1) in one of the two planes, and 2) in the intersection of the two planes.

12. If the centre of perspective be a point Σ_1 in the plane Π_1, and if σ' be the line in which the second plane Π' is intersected by Π_1, it is obvious, *first*, that to every point M' of Π', which is situated on σ', corresponds, in Π_1, an indeterminate point on the line $\overline{\Sigma_1 M'}$, whilst to every other point of Π' corresponds the point Σ_1 itself; and, *secondly*, that to the point Σ_1 corresponds a wholly indeterminate point of the plane Π', whilst to every other point M_1 of Π_1 corresponds the intersection M' of $\overline{\Sigma_1 M_1}$ and σ'.

With respect to the correspondence between the lines of the two planes, it is equally obvious, *first*, that to the line σ' in Π' corresponds a wholly indeterminate line in Π_1, whilst to every other line m' in Π' corresponds the line through Σ_1 and $(m'\sigma')$; and, *secondly*, that to every line m_1 in Π_1, which passes through Σ_1, corresponds an indeterminate line in Π' passing through the intersection $(m_1\sigma')$, whilst to every other line in Π_1 corresponds the line σ' itself.

13. If the centre of perspective Σ_1 lie in the intersection σ' of the two

* "Comptes rendus d s séances de l'Académie des Sciences," 1864.

planes, Π_1 and Π', then to Σ_1, regarded as a point of either plane, corresponds a wholly indeterminate point of the other plane; to any other point of σ', in either plane, corresponds an indeterminte point of the same line σ' in the other plane; whilst to every other point, of either plane, corresponds the point Σ_1 itself.

The lines of the two planes correspond in the following manner: to σ', regarded as a line of either plane, corresponds a wholly indeterminate line in the other plane; to any other line through Σ_1, in either plane, corresponds an indeterminate line through Σ_1 in the other plane; whilst to any other line, in either plane, corresponds the line σ' itself.

14. Passing now from the perspective, to any other position of the two planes, we conclude that the *homographic correspondence between them may assume either of the following two exceptional or singular forms.*

First. There may be a singular point in one plane, and a singular line in the other, whose respective correspondents are wholly indeterminate. To each point in the singular line will then correspond an indeterminate point in a determinate line passing through the singular point, whilst to each such line through the singular point will correspond an indeterminate line passing through a determinate point of the singular line.

It is of importance to observe that between the above-mentioned points on the singular line and the associated lines through the singular point there must, by art. 12, always exist an equi-anharmonic, or (1, 1) correspondence.

Secondly. In each plane there may be a singular line, and a singular point situated in that line, whose respective correspondents are wholly indeterminate. To every other point in a singular line will then correspond an indeterminate point in the other singular line, whilst to every other line passing through a singular point will correspond an indeterminate line through the other singular point.

15. If an exceptional homographic relation exist between two planes Π_1 and Π', and any ordinary correlation, such as that described in art. 3, be established between the latter, and a third plane Π_2, it is evident that between the former Π_1 and this third plane Π_2 an exceptional correlation will exist; and *vice versâ*, from any exceptional correlation, we can always pass, by means of an auxiliary ordinary correlation, to an exceptional homographic relation. This principle enables us, readily, to deduce *all possible exceptional forms of correlation* from the results of the last article.

16. In so doing we must consider three cases, since the first case of the last article presents two varieties.

First. The exceptional homographic relation between Π_1 and Π' is such that in Π_1 there is a singular point Σ_1, and in Π' a singular line σ'. The pole of σ', in the plane Π_2, being Σ_2, we shall have, between Π_1 and Π_2, an *exceptional correlation with singular points* Σ_1 and Σ_2, whose characteristic properties, easily deducible from art. 14, may be briefly described thus:

The polar of a singular point is wholly indeterminate. The pole of every

line through a singular point is an indeterminate point on a determinate line through the other singular point, the two lines thus associated being always corresponding rays of equi-anharmonic pencils.

Secondly. The exceptional homographic relation between Π_1 and Π' is such, that there is a singular line σ_1 in Π_1, and a singular point Σ' in Π'.

The polar of Σ', in the plane Π_2, being σ_2, we shall now have, between the planes Π_1 and Π_2, *an exceptional correlation with singular lines* σ_1 and σ_2, the characteristic properties of which, as deduced from art. 14, will be as follows :

The pole of a singular line is wholly indeterminate. The polar of every point in a singular line is an indeterminate line through a determinate point of the other singular line, the two points thus associated being always corresponding points of two equi-anharmonic rows.

Thirdly. The exceptional homographic relation between Π_1 and Π' is such that in each of these planes there is a singular line, and a singular point situated on that line.

Between the planes Π_1 and Π_2 we shall now have an *exceptional correlation with singular lines and points* (in each line a point), of which the following properties are characteristic :

The pole of each singular line, as well as the polar of each singular point, is wholly indeterminate. The polar of any point in a singular line, not coincident with the singular point situated therein, is an indeterminate line through the other singular point.

An exceptional correlation with singular points is determined when the positions of those points, and of three pairs of conjugate lines passing through them are known. In like manner, an exceptional correlation with singular lines requires, for its determination, a knowledge of the positions of those lines, as well as of three pairs of conjugate points situated thereon. Seven arbitrary conditions, of the kind described in art. 5, are necessary and sufficient, as we shall see, to determine an exceptional correlation with singular points, or with singular lines. An exceptional correlation of the third kind, however, that is to say, one which possesses singular lines and singular points situated therein, cannot, in general, be made to satisfy more than six such conditions. Hence it is that such exceptional correlations do not present themselves in the present paper; they have been described, solely, for the sake of completeness.*

* It is well known, and has recently been again demonstrated by Schröter, ("Journal für die reine und angewandte Mathematik," vol. 77, p. 105,) that two correlated planes can always be made to coincide in such a manner, that to each point the same line shall correspond, no matter to which of the two coincident planes that point be ascribed. The point and line, in fact, then become pole and polar relative to a conic. Now this conic degenerates to a line-pair, when the correlation has singular points ; to a point-pair, when it has singular lines ; and to a line-pair-point, when it has singular points and lines. Hence the connection between degenerate conics and exceptional correlations, alluded to in art. 10.

17. The following properties of the two first kinds of exceptional correlations will be frequently referred to; they are easily deduced from those already described.

a. The pole of every line, in either plane, which does not pass through the singular point of that plane, coincides with the singular point of the other plane.

b. The pole of every point, in either plane, which does not coincide with the singular point of that plane, is a determinate line passing through the singular point of the other plane.

c. The polar of every point, in either plane, not situated on a singular line of that plane, coincides with the singular line of the other plane.

d. The pole of every line, in either plane, which does not coincide with the singular line in that plane, is a determinate point in the singular line of the other plane.

Relations between the Characteristics and Singularities of any System of Correlations.

18. M_1 being an arbitrary point in one of two correlated planes, its polars, in the several correlations of any system, envelope a curve, whose class μ indicates the number of correlations of the system, in each of which the polar of M_1 passes through an arbitrary point M_2 of the second plane. But since, in each of these correlations, (and in these solely) the polar of M_2 will pass through M_1 (art. 4), μ will also be the class of the curve enveloped by the polars of an arbitrary point M_2 in the second plane.

Regarded as the number of correlations of the system, in each of which two arbitrary points M_1 and M_2 are conjugate to each other, μ may be appropriately termed the *class of the system of correlations.*

19. In like manner, the poles of an arbitrary line, in either plane, lie on a curve whose order ν may be termed *the order of the system of correlations*, since it indicates the number of correlations of the system in each of which two arbitrary lines m_1 and m_2 are conjugate to each other.

20. Between the *characteristics* μ and ν of a system of correlations, and the number of exceptional correlations which that system includes, relations exist precisely like those which have been established by Chasles between the characteristics of a system of conics satisfying four given conditions, and the number of degenerate conics included in the system.

In fact, if π denote the number of exceptional correlations with singular points, and λ the number of exceptional correlations with singular lines included in the system of correlations whose characteristics are

μ and ν, then $$\left.\begin{aligned}\mu &= 2\nu-\pi\\ \nu &= 2\mu-\lambda\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Hence we deduce $$\left.\begin{aligned}\lambda+\pi &= \mu+\nu\\ \lambda-\pi &= 3\,(\mu-\nu)\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

and also $$\left.\begin{aligned}3\mu &= 2\lambda+\pi\\ 3\nu &= \lambda+2\pi\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

21. Since each of the relations (1) follows from the other by the Principle of Duality, a demonstration of the first will be sufficient.

If a_1 and b_1 be any two lines in the first plane, and M_2 any point in the second plane, we have to determine the number of correlations of the system (μ, ν) in each of which the polar $\overline{A_2B_2}$ of the intersection (a_1b_1) passes through M_2. By hypothesis, there are ν correlations of the system in each of which the pole A_2 of a_1 lies in an arbitrary ray m_2 of the pencil whose centre is M_2. If the pole B_2 of b_1, in each of these ν correlations, be joined to M_2 by a ray m_2', then to each ray m_2 will correspond ν rays m_2'.

In like manner, to each ray m_2' will correspond ν rays m_2, passing, respectively, through the poles of a_1 in the several correlations in which the poles of b_1 lie in m_2'. Hence there is a (ν, ν) correspondence between the rays m_2 and m_2', and consequently there are 2ν rays, with each of which an m_2 and its corresponding m_2' coincide.

Although each of these 2ν rays passes through the poles of a_1 and b_1 in one and the same correlation, they are not all polars of (a_1b_1). For instance, if Σ_1 and Σ_2 be the singular points of one of the π exceptional correlations included in the system, the poles of the *arbitrary* lines a_1 and b_1 will coincide with Σ_2 (art. 17. a.), and by so doing cause m_2 and m_2' to coincide with $\overline{M_2\Sigma_2}$. The polar of (a_1b_1) in this exceptional correlation, however, will not in general coincide with this *arbitrary* line $\overline{M_2\Sigma_2}$, but with that ray of the pencil (Σ_2) which corresponds equianharmonically with the ray, through (a_1b_1), of the pencil (Σ_1), (arts. 16. first case, and 17. **b**.)

Exceptional correlations with singular points being, clearly, the only ones in which the poles of a_1 and b_1 coincide, it follows, as indicated by the first of the relations (1) in art. 20, that the number μ of correlations in which the polar of (a_1b_1) passes through the arbitrary point M_2 is less than 2ν by the number π of exceptional correlations in the system which possess singular points.

22. The number of exceptional correlations in a system satisfying seven given conditions being directly determined, the relations (1) of art. 20, written in the form (3), will enable us to deduce at once the characteristics of the system.

I propose to apply this method to the determination of the characteristics of the *fundamental systems*, that is to say, of those systems which satisfy elementary conditions of the following types :

A given point shall have a given polar (2 conditions).
A given line shall have a given pole (2 conditions).
Two given points shall be conjugate (1 condition).
Two given lines shall be conjugate (1 condition).

The characteristics of the fundamental systems being thus determined, it will be easy to deduce the number of correlations which satisfy any eight given elementary conditions.

All that will be then requisite in order to enable us to ascend from elementary conditions to others of a more complex character, and thus to solve the problem of the correlation of two planes in all its generality, will be a more intimate knowledge of the properties of the curves of the order ν, and of the curves of the class μ which, as we have seen (arts. 18 and 19), are associated with each system of correlations.

Enumeration and Classification of the Fundamental Systems of Correlations.

23. α, β, γ, δ being integers satisfying the condition

$$2\alpha+2\beta+\gamma+\delta=7,$$

we shall term $(\alpha\,\beta\,\gamma\,\delta)$ the *signature* of the system of correlations satisfying the following conditions:

α points in one plane have given polars in the other.
β right lines in the first plane have given poles in the second.
γ points and δ lines in each plane have given conjugates in the other plane.

It is obvious that the systems of correlations whose signatures are $(\alpha\,\beta\,\gamma\,\delta)$ and $(\beta\,\alpha\,\gamma\,\delta)$ are identical. The first symbol, in fact, being regarded as descriptive of the data in the one plane, the second will indicate the corresponding data in the correlated plane.

Now the total number of integral solutions of the equation

$$2\alpha+2\beta+\gamma+\delta=7$$

can be readily proved to be 52; hence *there are twenty-six distinct fundamental systems of correlations.*

24. These systems may be arranged in six groups as follows:

Group.	Signature.	
I.	(3010)	(0301)
II.	(2110)	(1201)
III.	(2030)	(0203)
	(2021)	(0212)
IV.	(1130)	(1103)
	(1121)	(1112)
V.	(1050)	(0105)
	(1041)	(0114)
	(1032)	(0123)

VI.	(0070)	(0007)
	(0061)	(0016)
	(0052)	(0025)
	(0043)	(0034)

25. In each group, the systems are arranged in two columns, such that the data of each system in one column are correlative to the data of the system opposite thereto in the other column. Now to every correlation of the system $(\alpha\,\beta\,\gamma\,\delta)$ will obviously correspond, by the Principle of Duality, a correlation of the system $(\beta\,\alpha\,\delta\,\gamma)$; moreover, the class and order of the first system will be equal, respectively, to the order and class of the second system; and for every exceptional correlation with singular points in the former there will be an exceptional correlation with singular lines in the latter, and *vice versâ*. It will suffice, therefore, to consider the systems contained in one only of the two preceding columns.

Number and Nature of Exceptional Correlations in the Fundamental Systems.

26. I proceed first to determine, directly, the number and nature of the exceptional correlations included in each of the thirteen fundamental systems whose signatures stand in the first column of the preceding article. In doing so, two associated singular points of any exceptional correlation will always be indicated by the symbols Σ_1 and Σ_2, and two associated singular lines by σ_1 and σ_2.

27. Throughout this investigation it will be assumed that, between the positions of the given points and lines, no special relation whatever exists.

28. Let the symbol of the system (3010) be (3010)

$$\begin{pmatrix} P_1 & Q_1 & R_1 & A_1 \\ p_2 & q_2 & r_2 & A_2 \end{pmatrix}.$$

In this system there can be no exceptional correlation with singular $\lambda = 0$
lines. To prove this, I observe, *first*, that σ_2 could not coincide with any one of the three lines $p_2\,q_2\,r_2$, say p_2; for if it did, then Q_1 and R_1, as poles of q_2 and r_2, would lie on σ_1 (art. 17. **d**.); and A_1 not being situated thereon (art. 27), would have σ_2 for its polar, whereas by hypothesis this polar should pass through A_2; and *secondly*, that if there were a singular line σ_2 not coincident with any one of the three lines $p_2\,q_2\,r_2$, then the poles $P_1\,Q_1\,R_1$ of these lines would all lie on σ_1 (art. 17. **d**.), which is inconsistent with the assumed generality of the data by which the system is defined (art. 27).

Again, if there be a singular point Σ_1 in the system, it must coincide $\pi=3$
with one of the three points $P_1\,Q_1\,R_1$; for if it did not, $p_2\,q_2\,r_2$ would concur in Σ_2 (art. 17. **b**.), which is not the case. Now if Σ_1 were coinci-

dent with P_1, its polar, being perfectly arbitrary (art. 16.), might obviously be regarded as coincident with p_2. Moreover, q_2 and r_2, as polars of Q_1 and R_1, would then intersect in Σ_2 (art. 17. **b**.), and the polar of A_1 would be $\overline{A_2\Sigma_2}$. In this manner all the seven conditions would be fulfilled, and the exceptional correlation would be perfectly and uniquely determined (art. 16) ; the polar of any point M_1, for instance, would be the ray m_2, which passes through $(q_2 r_2)$ and satisfies the anharmonic relation $$P_1(Q_1 R_1 A_1 M_1) = (q_2 r_2)(q_2 r_2 A_2 m_2).$$

In like manner, Σ_1 might coincide with Q_1, if Σ_2 were coincident with $(r_2 p_2)$; or, lastly, Σ_1 might coincide with R_1, if Σ_2 were coincident with $(p_2 q_2)$. Hence we conclude that *in the system under consideration there are but three exceptional correlations, and that each of these has singular points.* We have, consequently, by art. 20. (3), the values

$$\lambda = 0, \quad \pi = 3, \quad \mu = 1, \quad \nu = 2.$$

(2110) 29. The next system (2110) may be denoted by

$$\begin{pmatrix} P_1 & Q_1 & r_1 & A_1 \\ p_2 & q_2 & R_2 & A_2 \end{pmatrix}.$$

$\lambda=1$ Here r_1 cannot be coincident with σ_1 ; for if it were, p_2 and q_2, as polars of points not situated on σ_1, would be coincident with σ_2 (art. 17. **c**.) ; hence R_2 must lie on σ_2 (art. 17. **d**.) But if so, then, by art. 27, neither p_2 nor q_2 can coincide with σ_2, and as a consequence, P_1 and Q_1 must lie on σ_1. Again A_1, not being on σ_1, its polar, which by hypothesis passes through A_2, must be coincident with σ_2. Hence the *only possible correlation with singular lines is that in which these lines coincide, respectively, with $\overline{P_1 Q_1}$ and $\overline{R_2 A_2}$*; that this correlation satisfies the seven given conditions, and is precisely determined by them, is readily seen.

$\pi=1$ Passing to possible exceptional correlations with singular points, it is obvious that R_2 cannot coincide with Σ_2 ; for if it did, P_1 and Q_1, the poles of two lines p_2 and q_2 which do not pass through R_2, would be *coincident* in Σ_1 (art. 17. **a**.), which is not the case. Hence we infer, by art. 17. **b**., that r_1 must pass through Σ_1, and as a consequence of this, —or rather of the fact that neither P_1 nor Q_1 can, under these circumstances, coincide with Σ_1,—that p_2 and q_2 must intersect in Σ_2.

That *there is one and only one exceptional correlation which has a singular point Σ_2 coincident with $(p_2 q_2)$, and its associate Σ_1 situated in r_1*, and that this correlation is precisely determined by the seven given conditions, is obvious on remarking that the anharmonic relation

$$\Sigma_1(P_1 Q_1 r_1 A_1) = (p_2 q_2)(p_2 q_2 R_2 A_2)$$

is the only remaining condition which Σ_1 has to fulfil (art. 16.). The point Σ_1, in fact, is the intersection, with r_1, of the line which connects A_1 with the *only* point A on $\overline{P_1 Q_1}$ which satisfies the relation

$$\overline{P_1 Q_1}(P_1 Q_1 r_1 A) = (p_2 q_2)(p_2 q_2 R_2 A_2).$$

For the system now under consideration, the equations (3) of art. 20 furnish the following values:—

$$\lambda = 1, \quad \pi = 1, \quad \mu = 1, \quad \nu = 1.$$

30. The next system to be considered has the signature (2030), and the symbol (2030)

$$\begin{pmatrix} P_1 & Q_1 & A_1 & B_1 & C_1 \\ p_2 & q_2 & A_2 & B_2 & C_2 \end{pmatrix}.$$

Neither p_2 nor q_2 can here coincide with σ_2; for if either did, the polars of $A_2B_2C_2$ would be coincident with σ_1 (art. 17. **c**.), and therefore could not pass, respectively, through $A_1B_1C_1$. Hence we infer that if σ_1 exist, P_1 and Q_1 must lie on it (art. 17. **d**.). But if σ_1 were coincident with $\overline{P_1Q_1}$, the polars of $A_1B_1C_1$ would be coincident with σ_2 (art. 17. **c**.), which is inconsistent with the condition of their passing, respectively, through $A_2B_2C_2$. Hence we conclude that *there are no exceptional correlations with singular lines in the system.* $\lambda=0$

Again, if there be a singular point Σ_2, it must lie in one, at least, of the two lines p_2 and q_2, for otherwise P_1 and Q_1 would be coincident in Σ_2 (art. 17. **a**.). If Σ_2 were on p_2, but not on q_2, its associate Σ_1 would coincide with Q_1 (art. 17.), and Σ_2 itself would, by art. 16., simply have to satisfy the condition $\pi=3$

$$Q_1\,(P_1A_1B_1C_1) = \Sigma_2\,(p_2A_2B_2C_2) \qquad \text{(1)}.$$

This is clearly possible, and in one way only (art. 29).

In like manner, there is a second exceptional correlation for which Σ_1 is coincident with P_1, and Σ_2 lies on q_2 so as to satisfy the condition

$$P_1\,(Q_1A_1B_1C_1) = \Sigma_2\,(q_2A_2B_2C_2) \qquad \text{(2)}.$$

In every other exceptional correlation, Σ_2 must be coincident with (p_2q_2), and Σ_1 must satisfy the homographic relation

$$\Sigma_1\,(P_1Q_1A_1B_1C_1) = (p_2q_2)\,(p_2q_2A_2B_2C_2).$$

Now it is well known that there is, in general, one and only one position for Σ_1 consistent with this relation.* It is, in fact, the fourth intersection of two conics (S_1') and (S_1'') circumscribed to the triangle $A_1B_1C_1$, one of which (S_1') passes through P_1, and is determined by the anharmonic relation

$$(S_1')\,(P_1A_1B_1C_1) = (p_2q_2)(p_2A_2B_2C_2),$$

and the other (S_1'') passes through Q_1 in such a manner that

$$(S_1'')\,(Q_1A_1B_1C_1) = (p_2q_2)\,(q_2A_2B_2C_2).$$

The only exceptional case which can present itself arises when the above two conics (S_1') (S_1'') happen to coincide. This case, however, is excluded by art. 27, since it implies the existence of a special relation between the positions of the given lines and points.

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* See Sturm's excellent paper on the *Problem der Projectivität und seine Anwendung auf die Flächen zweiten Grades*, in the "Mathematische Annalen," vol. i. p. 533, wherein many of the auxiliary theorems employed in this paper are elaborately demonstrated.

We conclude, then, that *the system under consideration contains three exceptional correlations with singular points*, and accordingly we have, as in art. 28, $\lambda = 0,\ \ \pi = 3,\ \ \mu = 1,\ \ \nu = 2.$

(2021) 31. We proceed to the system (2021) whose symbol is

$$\begin{pmatrix} P_1 & Q_1 & A_1 & B_1 & c_1 \\ p_2 & q_2 & A_2 & B_2 & c_2 \end{pmatrix}.$$

$\lambda=1$ If p_2 were coincident with a singular line σ_2, then Q_1 would lie on σ_1 (art. 17. **d**.), and at the same time the polars of A_2 and B_2 would coincide with σ_1 (art. 17. **c**.). But, since Q_1, A_1 and B_1 are not in a line (art. 27), this is obviously inconsistent with the condition that these polars should pass, respectively, through A_1 and B_1. In like manner, q_2 cannot be coincident with σ_2; hence, if this singular line exist at all, P_1 and Q_1 must lie on its associate σ_1 (art. 17. **d**.), and the polars of A_1 and B_1 must coincide with σ_2 (art. 17. **c**.), that is to say, σ_2 must pass through A_2 and B_2.

This being clearly possible, and moreover the seven conditions being precisely sufficient to determine such an exceptional correlation,* we conclude that *the system contains one, and only one, exceptional correlation with singular lines;* the latter being coincident, respectively, with $\overline{P_1Q_1}$ and $\overline{A_2B_2}$.

$\pi=4$ Again, it can be shown, as in art. 30, that if there be a singular point Σ_2, it must lie on one, at least, of the lines $p_2\,q_2$. Now if Σ_2 were situated on p_2, but not on q_2, then its associate Σ_1 would be coincident with Q_1, and the pole of c_1 would be coincident with Σ_2 (art. 17. **a**.); but as this pole must be situated on c_2, (p_2c_2) is the only possible position for Σ_2. In like manner, if Σ_2 were on q_2, but not on p_2, it would necessarily coincide with $(q_2\,c_2)$, and its associate Σ_1 with P_1. Moreover, it is evident that the seven given conditions suffice precisely to determine one of each of the exceptional correlations here described.

The only remaining position possible for Σ_2 is $(p_2\,q_2)$, in which case the polar of c_2, necessarily a point on c_1, would be coincident with Σ_1 (art. 17. **a**). The sole condition to be satisfied by this point Σ_1 on c_1 is

$$\Sigma_1\,(P_1\ Q_1\ A_1\ B_1) = (p_2q_2)\,(p_2\ q_2\ A_2\ B_2) \qquad \ldots\ldots\ldots\ldots\ldots (1);$$

hence Σ_1 must be one of the intersections of c_1 by the conic (S_1) which passes through $P_1\ Q_1\ A_1\ B_1$ and satisfies the anharmonic relation

$$(S_1)\,(P_1\ Q_1\ A_1\ B_1) = (p_2q_2)\,(p_2\ q_2\ A_2\ B_2).$$

Taking into account the two solutions of (1), we conclude that *in the system under consideration there are four exceptional correlations with singular points.*

* The pole of any line m_1, for example, would be the point M_2, on $\overline{A_2B_2}$, determined by the relation $\overline{P_1Q_1}\,(P_1\,Q_1\,c_1\,m_1) = \overline{A_2B_2}\,(p_2\,q_2\,c_2\,M_2)$.

The equations (3) of art. 20 give us, in the present case, the values

$$\lambda = 1, \quad \pi = 4, \quad \mu = 2, \quad \nu = 3.$$

32. We arrive next at the system (1130), which is defined by the symbol (1130)

$$\begin{pmatrix} P_1 & p_1 & A_1 & B_1 & C_1 \\ p_2 & P_2 & A_2 & B_2 & C_2 \end{pmatrix}.$$

Since p_1, in this case, cannot coincide with σ_1, (for if it did, the polars of $P_1 A_1 B_1 C_1$ would be coincident with σ_2, which is obviously impossible,) σ_2, if it exist, must pass through its pole P_2, and for a like reason its associate σ_1 must pass through P_1 (art. 17. d.). Such an exceptional correlation, however, could not possibly satisfy the conditions in virtue of which the polars of A_1, B_1, C_1 pass, respectively, through A_2, B_2, C_2; hence we conclude that the *system under consideration includes no exceptional correlation with singular lines.* $\lambda=0$

If P_1 were a singular point, P_2, as pole of a line p_1 which does not pass through that singular point, would necessarily be its associate, and *vice versâ* (art. 17. a.). Such a correlation is clearly possible, and in one way only. $\pi=3$

In any other exceptional correlation with singular points, Σ_1 would necessarily lie on p_1, since Σ_2 does not coincide with P_2; and for a like reason its associate Σ_2 would necessarily lie on p_2 (art. 17. b.). The sole condition to be fulfilled by these associated singular points Σ_1, Σ_2 is

$$\Sigma_1 (p_1 P_1 A_1 B_1 C_1) = \Sigma_2 (P_2 p_2 A_2 B_2 C_2) \quad \ldots\ldots\ldots\ldots\ldots(1),$$

a relation which can be satisfied in two, and only two, ways. To prove this, I observe that, as shown in art. 29, to each point S_1 on p_1 corresponds one, and only one, point S_2 on p_2 such that

$$S_1 (p_1 P_1 A_1 B_1) = S_2 (P_2 p_2 A_2 B_2);$$

whilst to this point S_2 corresponds one, and only one, point S_1' on p_1 such that

$$S_1' (p_1 P_1 A_1 C_1) = S_2 (P_2 p_2 A_2 C_2).$$

Hence we may say that to each point S_1 corresponds one, and only one, point S_1'; for a like reason, to each point S_1' corresponds one, and only one, point S_1. By a well-known theorem, therefore, there are two points on p_1 in each of which S_1 and S_1' coincide.

Each of these points, considered as a position of Σ_1, together with its corresponding S_2, considered as a position of Σ_2, will clearly give a solution of the relation (1). Hence we conclude that the *system under consideration includes three exceptional correlations with singular points.* For the system now under consideration, therefore, we have, as in arts. 28 and 30, the following values:

$$\lambda = 0, \quad \pi = 3, \quad \mu = 1, \quad \nu = 2.$$

33. Next, in the order of sequence adopted in art. 24, comes the system (1121), whose symbol is (1121)

$$\begin{pmatrix} P_1 & p_1 & A_1 & B_1 & c_1 \\ p_2 & P_2 & A_2 & B_2 & c_2 \end{pmatrix}.$$

$\lambda=2$ It can be shown here, precisely as in the preceding case, that if there be an exceptional correlation with singular lines, σ_1 must pass through P_1, and its associate σ_2 through P_2. But σ_1 must also pass either through A_1 or B_1; for if it did not do so, the polars of these points would coincide with σ_2 (art. 17. **c**.), which is impossible, because σ_2 could not pass at one and the same time through P_2, A_2, and B_2 (art. 27). It is obvious, however, that $\overline{P_1A_1}$ and $\overline{P_2B_2}$, as well as $\overline{P_1B_1}$ and $\overline{P_2A_2}$, will be associated singular lines of an exceptional correlation satisfying the given conditions. Hence the *system includes two, and only two, exceptional correlations with singular lines.*

$\pi=2$ Again, P_1 cannot coincide with a singular point Σ_1; for if it did, the pole of c_1 would coincide, in the associated singular point Σ_2, with P_2, the pole of p_1 (art. 17. **a**.); whereas, by hypothesis, the pole of c_1 lies on c_2. Hence Σ_2, if it exist, must lie on p_2 (art. 17. **b**.), and for a like reason its associate Σ_1 must lie on p_1. Moreover, if Σ_1 were not on c_1, its associate Σ_2 would, as pole of c_1 (art. 17. **a**.), lie on c_2; and *vice versâ*, if Σ_2 were not on c_2, its associate Σ_1 would lie on c_1. Hence we conclude that the *system includes two exceptional correlations with singular points.* In one of these Σ_1 is at (p_1c_1), and its associate Σ_2 is the sole point on p_2 which satisfies the relation

$$(p_1c_1)(p_1P_1A_1B_1) = \Sigma_2(P_2p_2A_2B_2);$$

in the other, Σ_2 is at (p_2c_2), and its associate Σ_1 is the sole point on p_1 which fulfils the condition

$$\Sigma_1(p_1P_1A_1B_1) = (p_2c_2)(P_2p_2A_2B_2).$$

In the present case, therefore, we have as result,

$$\lambda = 2, \quad \pi = 2, \quad \mu = 2, \quad \nu = 2.$$

(1050) 34. We have now reached the fifth group of fundamental systems of correlations, the first in which is the system (1050) defined by the symbol

$$\begin{pmatrix} P_1 & A_1 & B_1 & C_1 & D_1 & E_1 \\ p_2 & A_2 & B_2 & C_2 & D_2 & E_2 \end{pmatrix}.$$

$\lambda=0$ Here *there can be no exceptional correlations with singular lines;* for since no such line σ_1 could pass through more than two of the five given points $A_1B_1C_1D_1E_1$ (art. 27), the polars of at least three of these points would be coincident in σ_2 (art. 17. **c**.), and therefore could not all pass through the points respectively conjugate to them.

$\pi=3$ On the other hand, it is obvious here, as in art. 30, that P_1 *would be the singular point of one exceptional correlation in the system,* its associate being the sole point Σ_2 in the second plane for which

$$P_1(A_1B_1C_1D_1E_1) = \Sigma_2(A_2B_2C_2D_2E_2).$$

If there be any singular point Σ_1, not coincident with P_1, then its associate Σ_2 must lie on p_2 (art. 17. **b**.), and by art. 16 the two points must satisfy the homographic relation

$$\Sigma_1(P_1A_1B_1C_1D_1E_1) = \Sigma_2(p_2A_2B_2C_2D_2E_2) \quad \ldots\ldots\ldots\ldots(1),$$

an equation of which there are two solutions. The proof of this depends upon the following Theorem:

S_2 *being any point on* p_2, *the locus of a point* S_1 *such that*

$$S_1(P_1A_1B_1C_1D_1) = S_2(p_2A_2B_2C_2D_2) \text{.......................} (2)$$

is a cubic which passes through $A_1B_1C_1D_1$, *and has a double point at* P_1. In fact, it has already been proved in art. 32, that on any line drawn through one of these four points, say A_1, there are, exclusive of the point A_1 which is arbitrary, two points S_1 which satisfy the relation (2); and A_1 itself obviously does so when S_2 has the position determined, as at the end of art. 29, by the relation

$$A_1(P_1B_1C_1D_1) = S_2(p_2B_2C_2D_2).$$

Moreover, P_1 is a double point of the cubic, because, as in art. 31 (1), there are two points S_2 on p_2 such that

$$P_1(A_1B_1C_1D_1) = S_2(A_2B_2C_2D_2).$$

In like manner, the locus of a point S_1' such that

$$S_1'(P_1A_1B_1C_1E_1) = S_2(p_2A_2B_2C_2E_2)$$

is another cubic having a double point at P_1, and also passing through $A_1B_1C_1E_1$.

Now, exclusive of the double point P_1 (which counts as four intersections), and the three points A_1, B_1, C_1, none of which leads to a solution of (1), these two cubics intersect in two points Σ_1, each of which, with its associated point Σ_2 on p_2 (determined by the relation

$$\Sigma_1\,(P_1\,A_1\,B_1\,C_1) = \Sigma_2\,(\,p_2\,A_2\,B_2\,C_2),$$

and, therefore, the same for both cubics) obviously satisfies the relation (1). We conclude, therefore, that *the system now under consideration includes three exceptional correlations with singular points.*

We have again, therefore, the values

$$\lambda = 0, \quad \pi = 3, \quad \mu = 1, \quad \nu = 2.$$

35. The second system in the fifth group of art. 24 has the signature (1041) and the symbol (1041)

$$\begin{pmatrix} P_1 & A_1 & B_1 & C_1 & D_1 & e_1 \\ p_2 & A_2 & B_2 & C_2 & D_2 & e_2 \end{pmatrix}.$$

By a process of reasoning similar to that employed in the last two articles, it may be readily shown that in the present system *there are no exceptional correlations which have singular lines.* $\lambda=0$

If any singular point Σ_1 were coincident with P_1, the pole of e_1 would coincide with its associate Σ_2 (art. 17. a.). Hence, and from art. 16, we conclude that the singular point Σ_2 associated with P_1 lies on e_2, and satisfies the relation $\quad P_1\,(A_1\,B_1\,C_1\,D_1) = \Sigma_2\,(A_2\,B_2\,C_2\,D_2).$ $\pi=6$

Of this there are obviously two solutions (art. 31), in other words, *there are two points* Σ_2 *on* e_2 *each of whose associates is coincident with* P_1.

Again, if there were any singular point Σ_1 not coincident with P_1, its associate Σ_2 would necessarily lie on p_2 (art. 17. **b**.), and satisfy the relation

$$\Sigma_1 (P_1 A_1 B_1 C_1 D_1) = \Sigma_2 (p_2 A_2 B_2 C_2 D_2).$$

Now all such points Σ_1, as we have proved in the preceding article, lie on a cubic S_1^3 which passes through $A_1 B_1 C_1 D_1$ and has a double point at P_1. Hence, and from the fact that Σ_2 must lie on e_2 whenever Σ_1 is not on e_1, and *vice versâ*, that Σ_1 must lie on e_1 whenever Σ_2 is not on e_2 (art. 17. **a**.), we conclude that there are *four, and only four, possible positions of* Σ_2 *on* p_2. In fact, in one of them Σ_2 coincides with (p_2e_2), and its associate Σ_1 is the sole point on S_1^3 which satisfies the relation

$$\Sigma_1 (P_1 A_1 B_1 C_1 D_1) = (p_2e_2)(p_2 A_2 B_2 C_2 D_2),$$

whilst, in its remaining three positions on the line p_2, Σ_2 corresponds, respectively, to the three intersections $\Sigma_1' \Sigma_1'' \Sigma_1'''$ of e_1 and the cubic (S_1^3).

On the whole, therefore, *there are in the system six exceptional correlations with singular points.* We have consequently the values,

$$\lambda = 0, \quad \pi = 6, \quad \mu = 2, \quad \nu = 4.$$

(1032) 36. The last of the systems we have to consider in Group V. has the signature (1032), and is defined by the symbol

$$\begin{pmatrix} P_1 & A_1 & B_1 & C_1 & d_1 & e_1 \\ p_2 & A_2 & B_2 & C_2 & d_2 & e_2 \end{pmatrix}.$$

$\lambda=3$ If there be any singular line σ_1 it must pass through P_1, for otherwise its associate σ_2 would coincide with p_2, and the polars of $A_2 B_2 C_2$ would be coincident with σ_1 (art. 17. **c**.), instead of passing, respectively, through $A_1 B_1 C_1$. Moreover, σ_1 must likewise pass through one of the points $A_1 B_1 C_1$, otherwise the polars of these points would be coincident with σ_2, instead of passing, respectively, through $A_2 B_2 C_2$. But if σ_1 pass through P_1 and one of the points $A_1 B_1 C_1$, its associate σ_2, with which the polars of the other two will coincide (art. 17. **c**.), must pass through the conjugates of the remaining two points. On the other hand, it is obvious that all the seven given conditions could be satisfied, and in one way only, by each exceptional correlation of the above kind.

Hence we conclude that in *the system there are three exceptional correlations with singular lines;* the latter being $\overline{P_1A_1}$ and $\overline{B_2C_2}$, $\overline{P_1B_1}$ and $\overline{C_2A_2}$, and $\overline{P_1C_1}$ and $\overline{A_2B_2}$, respectively.

$\pi=6$ If P_1 were a singular point, (d_2e_2) would be its associate, since the poles of d_1 and e_1 would necessarily coincide with this associate (art. 17. **a**.), and, at the same time, be situated on d_2 and e_2 respectively. Moreover, it is obvious that one, and only one, exceptional correlation, with these singular points satisfies the required conditions.

Since, in any other possible correlation with singular points, P_1 would not coincide with Σ_1, Σ_2 would necessarily lie on p_2 (art. 17. **b**.).

Further, if Σ_2 were not coincident with either of the points in which p_2 is intersected by d_2 and e_2, then the poles of the latter lines would coincide, in $(d_1 e_1)$, with Σ_1 (art. 17. **a**.). This is clearly possible, and in one way only (art. 29); the necessary and sufficient relation to be satisfied by Σ_2 being $$(d_1 e_1)(P_1 A_1 B_1 C_1) = \Sigma_2 (p_2 A_2 B_2 C_2).$$

If Σ_2 were coincident with $(p_2 d_2)$, then Σ_1, as pole of e_2, would be situated on e_1, and satisfy the equation

$$\Sigma_1 (P_1 A_1 B_1 C_1) = (p_2 d_2)(p_2 A_2 B_2 C_2);$$

of this equation there are obviously two solutions (art. 31).

In like manner, if Σ_2 were coincident with $(p_2 e_2)$, then there would be two positions of Σ_1, on d_2, each of which would satisfy the necessary and sufficient condition

$$\Sigma_1 (P_1 A_1 B_1 C_1) = (p_2 e_2)(p_2 A_2 B_2 C_2).$$

We conclude, therefore, that *the system under consideration contains, on the whole, six exceptional correlations with singular points.*

The following are the numerical values deducible from the above results: $\lambda = 3, \quad \pi = 6, \quad \mu = 4, \quad \nu = 5.$

37. The first of the four systems in Group VI. has the signature (0070), and is thus defined: (0070)

$$\begin{pmatrix} A_1 & B_1 & C_1 & D_1 & E_1 & F_1 & G_1 \\ A_2 & B_2 & C_2 & D_2 & E_2 & F_2 & G_2 \end{pmatrix}.$$

There are here, as in art. 34, *no correlations with singular lines.* $\lambda = 0$

The necessary and sufficient conditions to be fulfilled by a pair of singular points is $\pi = 3$

$$\Sigma_1 (A_1 B_1 C_1 D_1 E_1 F_1 G_1) = \Sigma_2 (A_2 B_2 C_2 D_2 E_2 F_2 G_2) \;\ldots\ldots\; (1),$$

of which equation, as Sturm and others have shown, *there are three solutions.* This important theorem results, in fact, from the following considerations: In art. 34 (1) it has been shown that, irrespective of F_2 (which is an arbitrary point), there are on any line f_2 passing through F_2, two points S_2, to each of which corresponds a point S_1, such that

$$S_1 (A_1 B_1 C_1 D_1 E_1 F_1) = S_2 (A_2 B_2 C_2 D_2 E_2 F_2) \;\ldots\ldots\ldots\; (2).$$

It is obvious, too, that F_2 itself counts as one such point; since one, and only one, point S_1 can be found to satisfy the condition

$$S_1 (A_1 B_1 C_1 D_1 E_1) = F_2 (A_2 B_2 C_2 D_2 E_2).$$

Hence, and from symmetry, we infer that the locus of the point S_2, to which another point S_1 can correspond so as to satisfy the condition (2), is a cubic passing through the six points $A_2 B_2 C_2 D_2 E_2 F_2$. The corresponding points S_1 lie, of course, on another cubic, passing through $A_1 B_1 C_1 D_1 E_1 F_1$, between whose points and those of the first cubic a (1, 1) correspondence obviously exists.

In like manner, the locus of each of the points S_1, S_2, which satisfy the relation $S_1 (A_1 B_1 C_1 D_1 E_1 G_1) = S_2 (A_2 B_2 C_2 D_2 E_2 G_2)$ ...... (3) is a cubic passing, respectively, through the six points $A_1 B_1 C_1 D_1 E_1 G_1$, and the six points $A_2 B_2 C_2 D_2 E_2 G_2$. Exclusive of the five points $A_2 B_2 C_2 D_2 E_2$ and of another point S_2, which may be termed the *satellite* of this group of five out of the seven points $A_2 B_2 C_2 D_2 E_2 F_2 G_2$, the above two cubic loci in the second plane intersect in three points Σ_2, which, with their associated points Σ_1, satisfy the relation (1).

The excluded point S_2 is that which corresponds to *every* point S_1 on the conic $(A_1B_1C_1D_1E_1)$ in such a manner that

$$S_1 (A_1 B_1 C_1 D_1 E_1) = S_2 (A_2 B_2 C_2 D_2 E_2) \text{..............} (4).$$

It is, obviously, the fourth intersection of two conics (S_2') (S_2'') circumscribed to the triangle $A_2B_2C_2$, one of which passes through D_2, and makes $(S_2') (A_2 B_2 C_2 D_2) = E_1 (A_1 B_1 C_1 D_1)$,

and the other passes through E_2, and makes

$$(S_2'') (A_2 B_2 C_2 E_2) = D_1 (A_1 B_1 C_1 E_1).$$

This point S_2 is excluded for the following reason: Each of the cubic loci, in the first plane, determined by (2) and (3), cuts the conic $(A_1B_1C_1D_1E_1)$ once again; but under the assumption in art. 27 these two points of intersection will not be coincident, although by (2) and (3) each will have S_2 for its correspondent. Hence, although S_2 is common to the two cubics in the second plane, it cannot be regarded as one of the points Σ_2 with which a point Σ_1 is so associated as to satisfy the relation (1). It is easy to see, however, that the satellite S_2, defined by (4), is the only one of the four intersections of the two cubics, in the second plane, which fails to lead to a solution of (1).

We conclude, therefore, that the *system under consideration contains three exceptional correlations with singular points;* accordingly, we have the values $\lambda = 0, \quad \pi = 3, \quad \mu = 1, \quad \nu = 2.$

(0061) 38. We proceed to the second system (0061) of Group VI., the symbol for which is $\begin{pmatrix} A_1 & B_1 & C_1 & D_1 & E_1 & F_1 & g_1 \\ A_2 & B_2 & C_2 & D_2 & E_2 & F_2 & g_2 \end{pmatrix}$.

$\lambda=0$ For the reasons stated in art. 34, *this system can contain no exceptional correlations with singular lines.*

$\pi=6$ If Σ_1, Σ_2 be the associated singular points of any exceptional correlation, the condition

$$\Sigma_1 (A_1 B_1 C_1 D_1 E_1 F_1) = \Sigma_2 (A_2 B_2 C_2 D_2 E_2 F_2) \text{ ....} (1)$$

must be satisfied. Hence, as we have seen in art. 37, Σ_1 and Σ_2 must be corresponding points on two cubics; one of which passes through $A_1 B_1 C_1 D_1 E_1 F_1$, as well as through the satellite of each of the six groups of five selected from these six points, and the other passes through

$A_2\, B_2\, C_2\, D_2\, E_2\, F_2$, as well as through the six satellites similarly connected therewith.

Besides satisfying the relation (1), however, it is to be remembered that if Σ_1 be not situated on g_1, then Σ_2 must lie on g_2; and further, that if Σ_2 be not on g_2, then Σ_1 must be on g_1, whence we conclude that *in the system there are six, and only six, exceptional correlations with singular points*, and that in three of these Σ_1 is on g_1, whilst in the other three Σ_2 is on g_2.

We have thus the following system of values:

$$\lambda = 0, \quad \pi = 6, \quad \mu = 2, \quad \nu = 4.$$

39. We pass now to the third system (0052) of the Group VI., the symbol for which is $\begin{pmatrix} A_1 & B_1 & C_1 & D_1 & E_1 & f_1 & g_1 \\ A_2 & B_2 & C_2 & D_2 & E_2 & f_2 & g_2 \end{pmatrix}$. (0052)

Here again there are no exceptional correlations with singular lines. $\lambda=0$

One condition to be satisfied by the singular points of every exceptional correlation which the system may contain is $\pi=12$

$$\Sigma_1 (A_1\, B_1\, C_1\, D_1\, E_1) = \Sigma_2 (A_2\, B_2\, C_2\, D_2\, E_2) \ldots\ldots\ldots\ldots\ldots (1),$$

in virtue of which to each point Σ_1 will correspond, in general, one, and only one, point Σ_2, and *vice versâ*.* Hence we infer that *there can be but one position for Σ_1 exterior to both the lines f_1 and g_1*, for in such a case its associate would necessarily coincide with $(f_2 g_2)$ (art. 17. a.). That there will be one such position of Σ_1, however, follows from the assumption in art. 27.

In like manner, *there will be one, and only one, exceptional correlation in which Σ_1 will be coincident with $(f_1 g_1)$, and its associate exterior both to f_2 and g_2.*

In all other cases, Σ_1 will lie on f_1, and therefore Σ_2 on g_2; or else Σ_1 will be on g_1, and Σ_2 on f_2. To decide how often the first of these cases will present itself, it will clearly suffice to determine the order of the locus of the points Σ_2 which, by the relation (1), correspond to the several points Σ_1 of the right line f_1.

To do this we will first enquire into the number of points in which any line a_2, passing through A_2, is intersected by the required locus. Now we have already proved that the locus of the point Σ_1 which satisfies the condition

$$\Sigma_1 (A_1\, B_1\, C_1\, D_1\, E_1) = \Sigma_2 (a_2\, B_2\, C_2\, D_2\, E_2)$$

is a cubic passing through $B_1\, C_1\, D_1\, E_1$ and having a double point at A_1 (art. 34). Hence we may say that, exclusive of the point A_2, which has a perfectly arbitrary position on a_2, there are three positions of Σ_2 on a_2, to which correspond three positions of Σ_1 on f_1, such that the relation (1) is satisfied.

* The only exceptions to this arise when Σ_1 (or Σ_2) coincides with one of the five points in its plane, or with the satellite S_1 (or S_2) of these five (art. 37).

It is obvious, however, that this relation will also be satisfied when Σ_2 coincides with A_2, and Σ_1 with either of the intersections of f_1 and the conic (S_1), through $B_1 C_1 D_1 E_1$, determined by the anharmonic relation

$$(S_1)(B_1 C_1 D_1 E_1) = A_2 (B_2 C_2 D_2 E_2).$$

Hence, and from symmetry, we conclude that *if Σ_1 describe any right line f_1, the locus of the point Σ_2, associated therewith by the relation* (1), *is a quintic which has a double point at each of the five points* $A_2 B_2 C_2 D_2 E_2$.

This quintic, *which has moreover a sixth double point* (at the satellite S_2 of the above five points) corresponding to the intersections of f_1 and the conic $(A_1B_1C_1D_1E_1)$, cuts g_2 in five points Σ_2, to each of which corresponds, by (1), a point Σ_1 situated on f_1; whence we infer *that there are five exceptional correlations in the system under consideration, of which one singular point lies in f_1, and its associate in g_2.* In like manner we conclude that *there are five exceptional correlations of which one singular point lies in g_1, and its associate in f_2.*

Altogether, therefore, *the system includes twelve exceptional correlations with singular points, but none with singular lines.* Accordingly we have the following numerical values:

$$\lambda = 0, \quad \pi = 12, \quad \mu = 4, \quad \nu = 8.$$

(0043) 40. We have now arrived at the last of the fundamental systems which need investigation. Its signature is (0043) and its symbol

$$\begin{pmatrix} A_1 & B_1 & C_1 & D_1 & e_1 & f_1 & g_1 \\ A_2 & B_2 & C_2 & D_2 & e_2 & f_2 & g_2 \end{pmatrix}.$$

$\lambda=6$ For reasons before stated (art. 36), every singular line σ_1 must pass through two of the four points $A_1B_1C_1D_1$, and its associate σ_2 must pass through the conjugates of the remaining two; moreover, it is easily seen that the seven given conditions can be satisfied, and that in one way only, by an exceptional correlation whose singular lines have any one of the six possible positions above indicated. Hence *in the system there are six correlations with singular lines.*

$\pi=12$ The associated singular points Σ_1, Σ_2 of every exceptional correlation in the system must, in the first place, satisfy the anharmonic relation

$$\Sigma_1 (A_1 B_1 C_1 D_1) = \Sigma_2 (A_2 B_2 C_2 D_2).$$

The position of one of the two points being known, therefore, we have at once a conic on which the other must be situated.

Now the point Σ_1 must lie in one, at least, of the three lines $e_1 f_1 g_1$, otherwise the poles of these lines would be coincident in Σ_2 (art. 17. a.), which is obviously inconsistent with the condition of their being situated, respectively, on e_2, f_2, and g_2. Again, if Σ_1 lie on one only of the three lines $e_1 f_1 g_1$, then its associate will necessarily coincide with the intersection of the conjugates of the other two, and Σ_1 itself will be

coincident with one of the two points in which the first line is intersected by a conic determined by (1). Lastly, if Σ_1 coincide with an intersection of any two of the lines $e_1 f_1 g_1$, then Σ_2 will necessarily coincide with one of the two points in which the conjugate of the third line is intersected by another conic, likewise determined by (1). In every possible case, therefore, one of the associated points Σ_1, Σ_2 must coincide with the intersection of two of three given lines, and the other must have one of two known positions on the conjugate of the third.

Taking all possible combinations into consideration, and remembering that each one of them leads to two exceptional correlations satisfying the seven conditions, we conclude that *the system under consideration contains twelve exceptional correlations with singular points.* We have consequently the following system of values:

$$\lambda = 6, \quad \pi = 12, \quad \mu = 8, \quad \nu = 10.$$

41. The class, order, and singularities of each of the thirteen fundamental systems, arranged in the left-hand column in art. 24, having now been determined, those of the remaining thirteen systems may, as stated in art. 25, be deduced by the Principle of Duality. The results may be thus tabulated:

Group.	Signature.	Characteristics.		Singularities.	
	$(\alpha\beta\gamma\delta)$	μ	ν	π	λ
I.	(3010)	1	2	3	0
	(0301)	2	1	0	3
II.	(2110)	1	1	1	1
	(1201)	1	1	1	1
III.	(2030)	1	2	3	0
	(2021)	2	3	4	1
	(0212)	3	2	1	4
	(0203)	2	1	0	3
IV.	(1130)	1	2	3	0
	(1121)	2	2	2	2
	(1112)	2	2	2	2
	(1103)	2	1	0	3
V.	(1050)	1	2	3	0
	(1041)	2	4	6	0
	(1032)	4	5	6	3
	(0123)	5	4	3	6
	(0114)	4	2	0	6
	(0105)	2	1	0	3

Group.	Signature. $(\alpha\beta\gamma\delta)$	Characteristics. μ	ν	Singularities. π	λ
VI.	(0070)	1	2	3	0
	(0061)	2	4	6	0
	(0052)	4	8	12	0
	(0043)	8	10	12	6
	(0034)	10	8	6	12
	(0025)	8	4	0	12
	(0016)	4	2	0	6
	(0007)	2	1	0	3*

Number of Correlations satisfying eight Elementary Conditions.

42. The number of correlations which satisfy any eight elementary conditions may be readily determined from the preceding Table. In fact, if we indicate this number by the symbol $[\alpha\beta\gamma\delta]$, where α, β, γ, δ have the same signification as before (art. 23), but now satisfy the equation

$$2\alpha+2\beta+\gamma+\delta = 8,$$

we shall clearly have $[\alpha\beta\gamma\delta]$ *equal to the class of the system of correlations* $(\alpha\beta\,\overline{\gamma-1}\,\delta)$, *as well as to the order of the system of correlations* $(\alpha\beta\gamma\,\overline{\delta-1})$. But the systems $(\alpha\beta\gamma\delta)$ and $(\beta\alpha\gamma\delta)$ being identical (art. 23), and the *class* and *order* of the system $(\alpha\beta\gamma\delta)$ being the same, respectively, as the *order* and *class* of the system $(\beta\alpha\delta\gamma)$ (art. 25), it follows that $[\alpha\beta\gamma\delta] = [\beta\alpha\gamma\delta] = [\alpha\beta\delta\gamma] = [\beta\alpha\delta\gamma]$, in other words, that γ and δ as well as α and β are interchangeable in the symbol $[\alpha\beta\gamma\delta]$. The following Table, therefore, gives the number

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* The meanings of the symbols being suitably modified, the results contained in this Table are at once applicable to the case of two homographic planes. For by the method employed in art. 15, we have, corresponding to every system of correlations $(\alpha\beta\gamma\delta)$ in two planes Π_1, Π_2, *a system of homographic relations* $(\alpha\beta\gamma\delta)$ in the planes Π_1, Π', which satisfy seven conditions of the following type: α points in the first plane correspond to given points in the second; β lines in the first plane correspond to given lines in the second; γ points in the first plane each correspond to a point on a given line in the second; and δ lines in the first plane each correspond to a line through a given point in the second. Every such system of homographic relations contains exceptional ones. In π of these there is a singular point in the first plane and a singular line in the second, whilst in λ others there is a singular line in the first plane and a singular point in the second (art. 14). The points which correspond, in the several homographic relations of a system, to a given point in the first plane, lie on a curve of the order μ in the second; those which correspond to a point in the second plane, however, lie on a curve of the order ν in the first. Similarly, the lines which correspond to a given line in the first plane envelope a curve of the class ν in the second, whilst those which correspond to a line in the second plane envelope a curve of the class μ in the first.

It is worth observing that if each point in the plane Π_1 be connected with every point in the plane Π' which corresponds thereto, in the several homographic relations of a system whose characteristics are μ and ν, we shall have, in space, a *complex of the degree* $\mu+\nu = \lambda+\pi$ [art. 20 (2)], of which Π_1 and Π' are singular planes. We have consequently twenty-six distinct complexes associated with the several fundamental systems of homographic relations satisfying seven conditions.

of solutions in all possible cases where the eight given conditions are of the elementary kind described in art. 22 :

$$[4000] = 1;$$
$$[3100] = 1;$$
$$[2200] = 0;$$
$$[3020] = 1,$$
$$[3011] = 2;$$
$$[2120] = 1,$$
$$[2111] = 1;$$
$$[2040] = 1,$$
$$[2031] = 2,$$
$$[2022] = 3;$$
$$[1140] = 1,$$
$$[1131] = 2,$$
$$[1122] = 2;$$
$$[1060] = 1,$$
$$[1051] = 2,$$
$$[1042] = 4,$$
$$[1033] = 5;$$
$$[0080] = 1,$$
$$[0071] = 2,$$
$$[0062] = 4,$$
$$[0053] = 8,$$
$$[0044] = 10.*$$

43. The only results in the above Table which cannot be obtained in the manner described in art. 42 are the first three. Of these, however the first is well known, and has already been alluded to in art. 3. The second differs in form only from the first ; for whenever the poles of three lines are given, as in the first case, the polars of three points (the intersections of the given lines) are always known, and *vice versâ.* With respect to the third result, it will be at once seen, on writing the eight conditions in full thus :

$$\begin{array}{cccc} P_1, & Q_1, & r_1, & s_1 \\ p_2, & q_2, & R_2, & S_2 \end{array}$$

that they cannot possibly be satisfied by any correlation unless the anharmonic ratios $(r_1 s_1)(P_1 Q_1 r_1 s_1)$ and $\overline{R_2 S_2}(p_2 q_2 R_2 S_2)$ are equal to one another. The assumption of any such equality, however, would be inconsistent with that already made in art. 27, in virtue of which the positions of the given points and lines are perfectly arbitrary.

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* This Table obviously gives, also, the number of ways in which two planes may be rendered homographic (or put in perspective with each other), so as to satisfy eight conditions of the kind described in the note to art. 41.

Connexes determined by Fundamental Systems of Correlations.

44. In art. 18 allusion was made to the series (doubly infinite) of curves of the class μ, and to the series of curves of the order ν, which every system of correlations, satisfying seven conditions, determines in each of the two planes; and in art. 22 it was observed that upon a more intimate knowledge of the properties of these curves must depend the solution of the general problem of correlation. Although unprepared, at present, to treat this wide subject exhaustively, I propose, before terminating the present paper, to indicate briefly the more salient features of a few of the simplest series of curves of the above kind.

45. Before commencing to do so, I may observe that, according to the terminology employed by Clebsch in a very suggestive posthumous paper recently published in the "Mathematische Annalen" (vol. 6, p. 203), each point of one of our two planes, and its polar in any correlation of a system, constitute an *element of a connex* of the class μ and order ν;* whilst each point in the other plane, and any one of its polars constitute an element of the *conjugate connex.* As illustrations, therefore, of the Theory originated by the eminent geometer, whose loss to science is so universally deplored, as well as on account of their application to the general problem of correlation, the well-defined conjugate connexes to which the several fundamental systems of correlations lead are well worthy of full investigation.†

46. Each curve of the class μ may be termed the *representative* of the point by whose polars it is enveloped, and each curve of the order ν the *representative* of the line of whose poles it is the locus. It is obvious that *every representative of a point touches all the singular lines in its plane, and every representative of a line passes through all the singular points in its plane.*

C nnex (12) 47. Each of the systems (3010), (2030), (1130), (1050), and (0070) leads to a pair of conjugate connexes of the first class and second order. In each of these five systems, a point in either plane is represented by a point in the other plane, (in other words, every point of the plane has its conjugate, exactly in the same manner as the given points have,) whilst every line is represented by a conic passing through the three fixed singular points of the three exceptional correlations which each system includes.

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* A connex $(\mu\nu)$, of the class μ and order ν, is defined by a given relation $(\xi, \eta, \zeta)^\mu (x, y, z)^\nu = 0$ between the coordinates x, y, z of a point, and the coordinates ξ, η, ζ of an associated right line.

† I may here mention that this investigation was in a tolerably advanced state eighteen months ago, when my studies were interrupted by other duties. Uncertainty as to when these studies may be resumed, has at length induced me, contrary to my original intention, to publish the present paper, before completing the whole enquiry.

We have here, in fact, the well-known case of the *Quadric Correspondence* of two planes, presented in its most general form; the three pairs of singular points being identical with the three pairs of Principal Points. The five systems above enumerated correspond, obviously, to the five different ways of determining a Quadric Correspondence by means of given Corresponding Points, Principal Points, and Principal Lines.*

48. The systems (0301), (0203), (1103), (0105), and (0007) all lead to connexes of the second class and first order. Every right line, in each of these systems, is represented by a right line; in other words, every line has its conjugate, in exactly the same sense as the given lines have. Every point, however, is represented by a conic touching the singular lines of the three exceptional correlations which each system includes. In short, we have here a correspondence established between two planes which is, simply, the correlative of the ordinary Quadric Correspondence.† Connex (21)

49. Of the remaining sixteen fundamental systems of correlations, by far the simplest are the two whose signatures are (2110) and (1201), both of which lead to one and the same kind of connex of the first class and first order. Connex (11)

In illustration of this I will consider briefly the first of the two systems. Its complete definition is given by the symbol

$$\begin{pmatrix} P_1 & Q_1 & r_1 & A_1 \\ p_2 & q_2 & R_2 & A_2 \end{pmatrix}$$

and, as shown in art. 29, it contains two exceptional correlations; of which one has the singular lines $\sigma_1 \equiv \overline{P_1 Q_1}$ and $\sigma_2 \equiv \overline{R_2 A_2}$, whilst the other has the singular points $\Sigma_2 \equiv (p_2 q_2)$, and Σ_1; the latter point being so situated, on r_1, that

$$\Sigma_1 (P_1 Q_1 r_1 A_1) = \Sigma_2 (p_2 q_2 R_2 A_2).$$

Since the characteristics are $\mu = 1$ and $\nu = 1$, it is obvious, *first*, that the polars of any point (say M_1) in any *two* correlations of the system (*e.g.*, in the two exceptional ones) determine, by their intersection, the point M_2, through which all the other polars of M_1 pass; and, *secondly*, that the poles of any right line m_1 in the two exceptional correlations determine the line m_2, upon which all the other poles of m_1 are situated. Now if M_1 be neither coincident with Σ_1 nor situated on σ_1, its polar in the exceptional correlation with singular lines will coincide with the

* See Reye on *Geometrische Verwandschaften zweiten Grades*, in the *Zeitschrift für Mathematik und Physik*, vol. xi., 1865.

† The fact that we have no distinctive name for this correspondence is doubtless due to the circumstance that no terms correlative to the very convenient ones *quadric*, *cubic*, &c. are yet in general use. A special case of the correspondence in question was briefly described in my paper "On the Quadric Inversion of Plane Curves," published in the Proceedings of the Royal Society, vol. xiv., 1865, and in the "Annali di Matematica pura ed applicata," tom. vii., Roma, 1865. It was also distinctly referred to in the paper by Reye above cited.

singular line σ_2 (17. **a**.), and its polar m_2, in the exceptional correlation with singular points, will pass through Σ_2 (17. **b**.), and be determined by the relation $$\Sigma_1 (P_1 Q_1 r_1 M_1) = \Sigma_2 (p_2 q_2 R_2 m_2).$$ Consequently, every such point M_1 is *represented*, in the connex, by a *point* $M_2 \equiv (m_2 \sigma_2)$ *on the singular line* σ_2. If M_1 were coincident with Σ_1, however, m_2 would be perfectly indeterminate (art. 16.), and its representative M_2 would, as a consequence, be *an indeterminate point on* σ_2. Lastly, if M_1 were situated on σ_1, then, although m_2 would be determined by (1), as before, the polar of M_1 in the exceptional correlation with singular lines would be an indeterminate line through $(m_2 \sigma_2)$ (art. 16.); and since the latter in one of its possible positions would be coincident with m_2, the representative of M_1 must be regarded as an *indeterminate point in* m_2.*

50. The peculiar *point by point* representation to which we are led by the system of correlations under consideration is such, therefore, that 1) the singular point of either plane is represented by an indeterminate point in the singular line of the other plane; 2) a point in the singular line of either plane is represented by an indeterminate point in a determinate line through the singular point of the other plane; 3) every other point in the first plane is represented by a perfectly determinate point in the singular line of the second plane. From similar considerations we infer that the *line by line* representation, to which the present system of correlations leads, may be briefly described thus: 1) a singular line is represented by an indeterminate line through a singular point; 2) a line through a singular point, by an indeterminate line through a determinate point on a singular line; 3) every other line, in either plane, is represented by a perfectly determinate line through a singular point.

51. It will be observed that the representation just described is not, even in an exceptional sense, homographic; since one of the characteristic properties of homographic correspondence is not fulfilled by it,—I mean that property, in virtue of which to a point in a line always corresponds a point in the corresponding line. The representation above indicated, however, as well as that termed homographic, is to be regarded as one of the forms to which a connex of the first class and first order may lead.

52. It is also worth observing that in the connex now under consideration the locus of the points which represent those situated in a given right line of either plane, breaks up into the representative of that line in the other plane, and the singular line of the latter plane;

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* It is obvious that, in all the correlations of the system, except that which has singular lines, the polars of a point on a singular line in either plane are *coincident* with a line which passes through the singular point of the other plane.

and further, that the envelope of the lines which represent those passing through a given point of either plane, breaks up into the representative of that point in the other plane, and the singular point of the latter plane.

We have here, in fact, the simplest case of a general theorem which holds for all the connexes determined by the fundamental systems of art. 41, and which may be thus enumerated:

*The curve, of the order ν, which represents any straight line, is always a constituent part of the envelope of the curves, of the class μ, which represent the several points of that line; and, vice versâ, the curve, of the class μ which represents any point, is always a constituent part of the envelope of the curves, of the order ν, which represent the several right lines passing through that point.** In proof of this, it will be sufficient to show that if A_1 and a_2 be pole and polar in any correlation of a system, then, in that same correlation, the tangent at A_1 to the representative of a_2, and the point of contact of a_2 with the representative of A_1, will also be pole and polar.† Now in the correlation immediately following the *first*, let a'_2 be the polar of A_1, and A'_1 the pole of a_2, in which case, of course, $(a_2\, a'_2)$ will be the pole of $\overline{A_1 A'_1}$. Then, ultimately, as the latter correlation approaches to identity with the first, $(a_2\, a'_2)$ becomes the point of contact of a_2 with the representative of A_1, and $\overline{A_1 A'_1}$ becomes the tangent at A_1 to the representative of a_2.

53. The connexes, of the second class and second order, to which the systems (1121) and (1112) give rise, are essentially the same in character. We may confine our attention, therefore, to the first system, which has been denoted in art. 33 by the symbol Conne (22)

$$\begin{pmatrix} P_1 & p_1 & A_1 & B_1 & c_1 \\ p_2 & P_2 & A_2 & B_2 & c_2 \end{pmatrix},$$

and found to include four exceptional correlations. One of these has the singular lines $\sigma'_1 \equiv \overline{P_1 A_1}$, $\sigma'_2 \equiv \overline{P_2 B_2}$; another the singular lines $\sigma''_1 \equiv \overline{P_1 B_1}$ and $\sigma''_2 \equiv \overline{P_2 A_2}$; a third has the singular points $\Sigma'_1 \equiv (p_1 c_1)$ and Σ'_2, the latter being so situated on p_2 that

$$(p_1\, c_1)\,(p_1\, P_1\, A_1\, B_1) = \Sigma'_2\,(P_2\, p_2\, A_2\, B_2)\,;$$

and the fourth has the singular points Σ''_1 and $\Sigma''_2 \equiv (p_2\, c_2)$, of which the former is so situated on p_1 that

$$\Sigma''_1\,(p_1\, P_1\, A_1\, B_1) = (p_2\, c_2)\,(P_2\, p_2\, A_2\, B_2).$$

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* In studying the properties of the various connexes, this theorem is of the greatest utility. It was to it, as a fruitful source of theorems on Envelopes, that I referred in a Note published in the *Educational Times* for December, 1871 (Reprint, vol. xvi., p. 99). The theorems demonstrated in that Note were special cases of more general ones, deducible from the connex treated of in art. 53.

† It is in consequence of this that the two connexes, to which every system of correlations leads, are *conjugate* in the manner defined by Clebsch, at p. 208 of the paper referred to in art. 45.

The conjugate connexes to which we are led by the present system are such that every point, in either plane, is represented by a conic which touches the two singular lines of the other plane, and every right line in the first plane is represented by a conic which passes through the two singular points of the second.

54. It should be observed, however, that *the conics which represent the several points of each singular line, degenerate to point-pairs.*

Thus, if M_1 be a point on the singular line σ'_1, its polar, in the exceptional correlation whose singular lines are σ'_1 and σ'_2, is an *indeterminate line* through the point M'_2, on σ'_2, which is determined by the relation

$$\sigma'_1\,(P_1\,p_1\,c_1\,M_1) = \sigma'_2\,(p_2\,P_2\,c_2\,M'_2)\ \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Hence we at once infer that the conic which represents M_1 breaks up into the point M'_1, and another point M_2. The latter point must be situated on σ''_2, because this line is the polar of M_1 in the exceptional correlation whose singular lines are σ''_1 and σ''_2 (17. **c**.), and it must also satisfy the relation

$$\sigma'_1\,(P_1\,p_1\,A_1\,M_1) = \sigma''_2\,(p_2\,P_2\,A_2\,M_2)\ \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

because the polars of M_1, in the correlation whose singular points are Σ'_1 and Σ'_2, as well as its polar in the correlation whose singular points are Σ''_1 and Σ''_2, obviously pass through the point M_2 thus determined. This point M_2, it will be observed, is conjugate to M_1 in precisely the same sense that A_2 is conjugate to A_1; in fact, we may say that *every point on a singular line of either plane has a conjugate point situated on the non-associated singular line of the other plane.*

The relations (1) and (2) also show that there is a (1, 1) correspondence between the points M_2 and M'_2 of a nature such that when one of them coincides with P_2, or falls upon p_2, the other does the same. Hence, by a well known theorem, we conclude that $\overline{M_2\,M'_2}$ always passes through a fixed point S'_2 on p_2. In like manner, it can be shown that every point N_1 on the singular line σ''_1 is represented by a point-pair, N_2 (on σ'_2) and N''_2 (on σ''_2), whose double tangent $\overline{N_2\,N''_2}$ passes through a fixed point S''_2 on p_2. It can be shown, moreover, that *the six points Σ'_2, Σ''_2, M_2, M'_2, N_2, N''_2 always lie on a conic;* and by the theorem of art. 52 we at once infer that *this is the conic which represents the arbitrary line $\overline{M_1\,N_1}$.*

55. In a precisely similar manner, the following properties may be established :—

Every line through a singular point of either plane has a conjugate line passing through the non-associated singular point of the other plane.

Every right line m_1 passing through the singular point Σ'_1 is represented by a line-pair m_2, m'_2, of which m'_2 passes through Σ'_2 and m_2 through

Σ_2'', *and the double point* $(m_2 m_2')$ *of this line-pair lies on a fixed line* s_2' *passing through* P_2.

Every line n_1 *passing through* Σ_1'' *is represented by a line-pair* $n_2 n_2''$, *of which* n_2'' *passes through* Σ_2'', *and* n_2 *through* Σ_2'; *and the double point* $(n_2 n_2'')$ *of this line-pair lies on a fixed line* s_2'' *passing through* P_2.

Every four such lines m_2, m_2', n_2, n_2'', *together with the singular lines* σ_2', σ_2'' *constitute six tangents of the conic which represents the arbitrary point* $(m_1 n_1)$.

56. The degenerate conics of the first plane which represent points and lines in the second, have precisely similar properties, that is to say,—

The points of the point-pairs lie on the singular lines σ_1', σ_1'', *whilst their double tangents envelope a point-pair* S_1', S_1'', *whose own double tangent is coincident with the line* p_1. *The lines of the line-pairs pass through the singular points* Σ_1', Σ_1'', *whilst their double points describe a line-pair* $s_1' s_1''$, *whose own double point is coincident with* P_1.

57. A degeneration such as that above described occurs in every fundamental system of correlations. We may in fact say, generally:

The curve of the class μ *which represents any point through which pass* κ *singular lines, degenerates to* κ *points, situated on the associated singular lines, and to a curve of the class* $\mu-\kappa$, *which touches the remaining* $\lambda-\kappa$ *singular lines.*

The curve of the order ν *which represents any line in which lie* κ *singular points, degenerates to* κ *lines through the associated singular points, and to a curve of the order* $\nu-\kappa$ *which passes through the remaining* $\pi-\kappa$ *singular points.*

58. Next in point of simplicity are the connexes determined by the systems of correlations of which (2021) and (2012) are the signatures. The first of these connexes is of the second class and third order, the second of the third class and second order.

The properties of the latter being at once deducible from those of the former by the Principle of Duality, we may confine our attention to the first of the above systems, the symbol for which is Connex (23)

$$\begin{pmatrix} P_1\ Q_1\ A_1\ B_1\ c_1 \\ p_2\ \ q_2\ A_2\ B_2\ c_2 \end{pmatrix}.$$

In art. 31 the existence was proved of a pair of singular lines $\overline{P_1 Q_1}$ and $\overline{A_2 B_2}$, and four pairs of singular points, viz., P_1 and $(q_2 c_2)$, Q_1 and $(p_2 c_2)$, Σ_1' and $(p_2 q_2)$, and Σ_1'' and $(p_2 q_2)$, where Σ_1', Σ_1'' are the intersections, with c_1, of the conic (S_1) which passes through $P_1 Q_1 A_1 B_1$ so as to make the anharmonic ratio of these four points on it equal to

$$(p_2 q_2)\,(p_2\, q_2\, A_2\, B_2).$$

59. The mode in which the conics and cubics representing special

points and lines degenerate, is very instructive. Instead of attempting an exhaustive discussion of the question, however, I must limit myself here to those cases of degeneration which most elucidate the general character of the representative curves.

Every point C_2 on the singular line $\overline{A_2B_2}$ is represented, as we have already stated in art. 57, by a point-pair C_1, C_1'. The point C_1' is so situated on the associated singular line $\overline{P_1Q_1}$ as to satisfy the condition

$$\overline{P_1Q_1}\,(P_1Q_1c_1C_1') = \overline{A_2B_2}\,(p_2q_2c_2C_2) \quad \ldots\ldots\ldots\ldots\ (1);$$

the polar of C_2, in the exceptional correlation whose singular lines are $\overline{P_1Q_1}$ and $\overline{A_2B_2}$ being an *indeterminate* line through C_1'. The point C_1, through which the polar of C_2 in every other correlation of the system passes, is situated on the conic (S_1) and fulfils the condition

$$(S_1)(P_1\,Q_1\,A_1\,B_1\,C_1) = (p_2\,q_2)(P_2\,q_2\,A_2\,B_2\,C_2) \quad \ldots\ldots\ (2),$$

as is at once obvious on considering the polars of C_2 in the two correlations whose singular points are Σ_1' and (p_2q_2), and Σ_1'' and (p_2q_2).

60. Between the points of the conic (S_1), and those of the line $\overline{A_2B_2}$, therefore, a (1, 1) correspondence is established, such that any two corresponding points $C_1\,C_2$ thereof are conjugate to each other in the same sense as $A_1\,A_2$ and $B_1\,B_2$ are by hypothesis. The system of correlations, in fact, would suffer no change if $C_1\,C_2$ were substituted in place of either of the latter pairs of conjugate points.

61. *The conic*, in the second plane, *which represents any arbitrary point*, say A_1, *of the conic* (S_1), *is a line-pair whose double point coincides with the corresponding point* A_2 *on* A_2B_2. This follows from the circumstance that, in the system now under consideration, there are *two* correlations for which the polar of A_1 coincides with an arbitrary line a_2 passing through A_2; in other words, there are two correlations which satisfy the eight conditions

$$\begin{array}{ccccc} P_1, & Q_1, & A_1, & B_1, & c_1 \\ p_2, & q_2, & a_2, & B_2, & c_2. \end{array}$$

In proof of this we need merely refer to the Table in art. 42, where it will be seen that [3011] = 2.

62. From the above we infer that *every line* x_2 *in the second plane is represented by a cubic* x_1^3 *which has a double point* C_1 *on the conic* (S_1). For if C_1 be the point on (S_1) which corresponds, by art. 60, to the point C_2 in which $\overline{A_2B_2}$ is intersected by x_2, then, as we have just seen, there are two correlations of the system for which the pole of x_2 is coincident with C_1. *The cubic* x_1^3 *obviously passes also through the singular points* P_1, Q_1, Σ_1', Σ_1'' *as well as through the point* C_1' *on* $\overline{P_1Q_1}$, *which*, by (1) of art. 59, *corresponds to* C_2 *on* $\overline{A_2B_2}$.

63. Lastly, since two of the singular points of the second plane coin-

cide in $(p_2 q_2)$ it is obvious that *every line* x_1 *in the first plane is represented by a cubic* x_2^3, *which has a double point at* $(p_2 q_2)$, *and passes, likewise, through the singular points* $(p_2 c_2)$, $(q_2 c_2)$, *as well as through the point* C_2, *on* $\overline{A_2 B_2}$, *which corresponds*, by (1) of art. 59, *to the point* C_1' *in which* x_1 *intersects* $\overline{P_1 Q_1}$, *and through the points* D_2 *and* E_2, on $\overline{A_2 B_2}$, *which correspond*, by (2) of art. 59, *to the points* D_1 *and* E_1 *in which* x_1 *intersects the conic* (S_1).

64. The remark in art. 61 is susceptible of generalisation in the form of the following useful Theorems, with the enunciation of which I will conclude the present paper :

Theorem (i.) :—*In a system of correlations* $(\alpha\beta\gamma\delta)$, *the curve, of the class* $[\alpha\beta(\gamma+1)\delta]$ (art. 42), *which represents either of two conjugate points* A_1, A_2, *breaks up into the other, together with a point on each of the singular lines associated with those which pass through the former. The multiplicity of* A_2 *on the representative of* A_1 *is* $[(\alpha+1)\beta(\gamma-1)\delta]$, *and that of* A_1 *on the representative of* A_2 *is* $[\alpha(\beta+1)(\gamma-1)\delta]$. *The number of singular lines which pass through* A_1 *is*

$$[\alpha\beta(\gamma+1)\delta]-[(\alpha+1)\beta(\gamma-1)\delta],$$

and the number of those which pass through A_2 *is*

$$[\alpha\beta(\gamma+1)\delta]-[\alpha(\beta+1)(\gamma-1)\delta].$$

Theorem (ii.) :—*In a system of correlations whose signature is* $(\alpha\beta\gamma\delta)$ *the curve, of the order* $[\alpha\beta\gamma(\delta+1)]$ (art. 42), *which represents either of two conjugate lines* a_1, a_2, *breaks up into the other, together with a line through each of the singular points associated with those situated on the former. The multiplicity of* a_2 *on the representative of* a_1 *is* $[\alpha(\beta+1)\gamma(\delta-1)]$, *and that of* a_1 *on the representative of* a_2 *is* $[(\alpha+1)\beta\gamma(\delta-1)]$. *The number of singular points situated on* a_1 *is*

$$[\alpha\beta\gamma(\delta+1)]-[\alpha(\beta+1)\gamma(\delta-1)],$$

and the number of those situated on a_2 *is*

$$[\alpha\beta\gamma(\delta+1)]-[(\alpha+1)\beta\gamma(\delta-1)].$$

May 2nd, 1874.